HW3 Solution

Monday, December 21, 2009

1:44 PM

Here are the results from my MATLAB code.

p1 = 0.030 (1a)
P[N_k = 30] = 0.073 (1c)
A = 78 (1d. ii)
Frequency of occurrence for {N_k = 30} = 0.078 (1d. iii)
Frequency of occurrence for {W_k > 2 mins} = 0.368 (1e)
P[W_k > 2 mins] = 0.368 (1f)
V = 29946 (3a)
D is a geometric r.v. with mean = 33.333 and parameter r = 0.970 (3b)
B = 5962 (3c i)
(3c ii)
The proportion of call requests that were blocked is B/V = 0.199
From Erlang B formula, the blocking probability is 0.200

① $T = 1{,}000$ hrs.
$\lambda = 30$ arrivals per hour
$n = 10^6$

(a) p_1 = the probability of exactly one arrival in a slot.
= the mean number of arrivals in a slot (because the random variable is Bernoulli.)

$$= \lambda \times \frac{T}{n} = 30 \times \frac{1000}{10^6} = \boxed{0.03}$$

(b) –

(c) Time is divided into $m = 1000$ non-overlapping intervals.

(i) Mean $= \mathbb{E}N_k = \lambda \times \underbrace{\frac{T}{m}} = 30 \times \frac{1000}{10^3} = \boxed{30}$

width of each interval

30 30

(ii), (iii) $P[N_k = 30] = e^{-30}\dfrac{30^{30}}{30!} = \boxed{0.073}$

(Recall that the pmf of a Poisson r.v. is given by

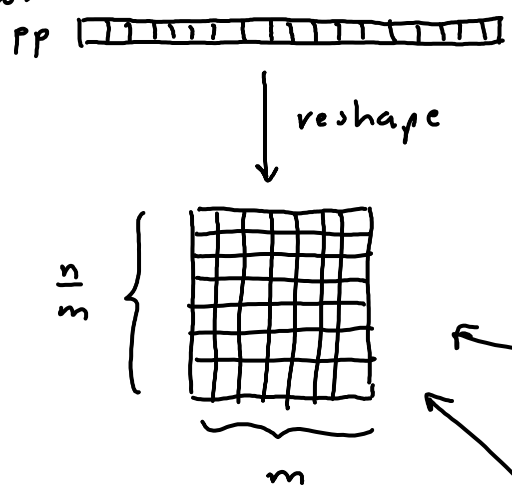
$$p_N(k) = P[N=k] = e^{-\alpha}\frac{\alpha^k}{k!}$$

where α is the mean.)

(iv) $P[N_k = 10.5] = \boxed{0.}$

Poisson r.v. only takes integer values.

(d) (i)



The command divide the pp (row vector) into m intervals.

Each piece becomes a column of .

The sum command adds up the 1's in the same column. So, the vector N consists of m member. The k^{th} member is the sum of values in the k^{th} column of . Because the 1's indicate arrivals, the k^{th} member of the vector N is N_k.

(ii) My MATLAB gives $A = 78$.

(iii) $\frac{A}{m} = 0.078$ which is close to the theoretical value of 0.073 in part (c.iii).

(e) 0.368 (MATLAB)

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(f) 0.368 (MATLAB) the same!

(2) (a) $f_X(x) = \frac{1}{\lambda} e^{-\lambda x}, \quad x > 0$

(b) $\int_a^b f_X(x)\,dx = \int_a^b \frac{1}{\mu} e^{-\mu x}\,dx = -e^{-\mu x}\Big|_a^b$

$$= e^{-a/\mu} - e^{-b/\mu}$$

(c) $a = (k-1)T$ and $b = kT$

$$\int_a^b f_X(x)\,dx = e^{-(k-1)T/\mu} - e^{-kT/\mu}$$

$$= e^{-(k-1)T/\mu}\left(1 - e^{-T/\mu}\right)$$

(d) $p_k = e^{-(k-1)T/\mu}\left(1 - e^{-T/\mu}\right), \quad k = 1, 2, 3, \ldots$

is Geometric with $r = e^{-T/\mu}$.

(3) (a) $v = 29946$ (MATLAB)

(b) (i) D is geometric : $p[D=k] = (1-r)r^{k-1}$

In this case, the T in (2) is replaced by $\frac{T}{n} = \frac{1000}{10^6} = \frac{1}{10^3}$ hr.

So, $r = e^{-\frac{T}{n}\mu} \approx 1 - \frac{T}{n}\mu$

Note that $\frac{1}{\mu} = 2\text{ min} = \frac{2}{60}$ hr.

$$= \frac{1}{30} \text{ hr.}$$

Therefore,

$$r = e^{-\frac{1}{1000}\times 30} \approx 0.97$$

(ii)

For geometric r.v. D with $p_D(k) = (1-r)r^{k-1}$

$\mathbb{E}D = \sum^{\infty} k(1-r)r^{k-1} = (1-r)\sum^{\infty} k\, r^{k-1}$

$$\mathbb{E}D = \sum_{k=1}^{\infty} k(1-r)r^{k-1} = (1-r)\sum_{k=1}^{\infty} k r^{k-1}$$

Recall that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Taking $\frac{d}{dr}$ on both sides we have

$$\sum_{k=1}^{\infty} k r^{k-1} = -\frac{1}{(1-r)^2}(-1) = \frac{1}{(1-r)^2}$$

$$\mathbb{E}D = (\cancel{1-r}) \times \frac{1}{(1-r)^{\cancel{2}}} = \frac{1}{1-r}$$

$$\approx \frac{1}{1-(1-\frac{I}{n}\mu)} = \frac{1}{\frac{I}{n}\mu} = \frac{n}{T} \times \frac{1}{\mu}$$

$$= 10^3 \times \frac{1}{30} \approx \boxed{33.3 \text{ slots}}$$

(c) (i) $B = 5962$ (MATLAB)

(ii) $\frac{B}{V} = \boxed{0.199}$ (MATLAB)

Note that $A = \frac{\lambda}{\mu} = 30 \times \frac{1}{30} = 1$

From Erlang B,

$$P_b = \frac{A^2/2!}{1 + A + \frac{A^2}{2}} = \frac{A^2}{2 + 2A + A^2} = \frac{1}{5}$$

$$= \boxed{0.2}$$

almost the same as simulation result.